

Anisotropic Pressures in Very Dense Magnetized Matter

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The problem of anisotropic pressures arising as a consequence of the spatial symmetry breaking introduced by an external magnetic field in quantum systems is discussed. The role of the conservation of energy and momentum of external fields as well as of systems providing boundary conditions in quantum statistics is considered. The vanishing of the average transverse momentum for an electron-positron system in its Landau ground state is shown, which means the vanishing of its transverse pressure. The situation for neutron case and Strange Quark Matter (SQM) in β -equilibrium is also briefly discussed. Thermodynamical relations in external fields as well as the form of the stress tensor in a quantum relativistic medium are also discussed. The ferromagnetic symmetry breaking is briefly discussed.

I. INTRODUCTION

The idea of local anisotropy in self gravitating systems is frequently discussed in the literature [1], although usually it is not studied how it is generated. However, the notion of local pressure anisotropy, that is, the occurrence of unequal principal stresses is a natural consequence of the spatial anisotropy introduced by external fields, as is evident to everybody to occur in atmospheric, earth crust and ocean pressures, where a preferred direction of increasing (radial) pressure is due to the (approximately) centrally-symmetric gravity force. This radial anisotropy is observed to occur, for instance, when considering a macroscopic sphere in any of these media. In a small neighborhood of any point inside it, the pressure seems to be isotropic, but actually this is not so, and it only remains unchanged as we move on isobar surfaces. But as we move perpendicularly to these surfaces, it changes due to the momentum added by the external gravity force field. A very well known anisotropy in pressures is present in rotating bodies, as planets and stars, since their mass is under the action of an axially symmetric centrifugal pressure which is added vectorially to the gravity force, and produces the flattening of these bodies.

For the case of a gas of electrically charged particles in an external constant homogeneous magnetic field B (which is field we will consider along this paper. For variable fields most of our results are valid for small volumes and/or short intervals of time), in classical electrodynamics, it is the Lorentz force $\mathbf{F} = e\mathbf{v} \times \mathbf{B}/c$ the source of an asymmetry in the pressures parallel and perpendicular to \mathbf{B} acting on the particles [2]. The Lorentz force stems from the fact that an electrically charged spinless particle, which in its interaction with the external magnetic field, moves in a way equivalent to an effective current, which in turn, generates a magnetic field opposite to B . That is the content of the Lenz law. By writing $e\mathbf{v} = \mathbf{j}\Delta V$, where $\Delta V = dx_1 dx_2 dx_3$, calling $f_i = F_i/\Delta V$ as the i -th component of the force density, and substituting $\mathbf{j} = c\nabla \times \mathcal{M}$, one has

$$f_i = -(\partial_i \mathcal{M}_s)B_s + (\partial_s \mathcal{M}_i)B_s \quad (1)$$

multiplying by $\Delta V = dx_1 dx_2 dx_3$ and assuming $B_s = B\delta_{s3}$ and $\partial \mathcal{M}_i/\partial x_3 = 0$ (actually it is also $\mathcal{M}_i = \mathcal{M}\delta_{i3}$), only

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the first term in (1) remains as nonzero and one gets back an expression for the force. For the pressure perpendicular to the field it results $p_{\perp} = -\mathcal{M} \cdot \mathbf{B}$. This is a classical effect and obviously p_{\perp} must be added to the usual kinetic isotropic pressure. As in classical electrodynamics \mathcal{M} is opposite to \mathbf{B} , then $\mathcal{M} \cdot \mathbf{B} < 0$, and $p_{\perp} > 0$. Its effect is similar to the centrifugal force mentioned above. However, the opposite case occurs when $\mathcal{M} > 0$ (i.e., in the paramagnetic or ferromagnetic cases) being oriented along \mathbf{B} , which occurs in the quantum case due to the coupling of the elementary magnetic moments due to spin with the external field. This quantum effect cannot be derived, obviously, from the classical Lorentz force.

After our papers [2], [3], claims were made against the results by some authors, as [4], [5]) starting from classical results. The contradiction stems from the fact that usually the classical stress tensor is derived from the Lorentz force [19]. But in a degenerate quantum gas the dynamics is described by the Dirac equation, and the spin coupling to the magnetic field plays a fundamental role (*that is, the particle does not obey the classical Lorentz force equation*), and the use of the classical stress tensor may lead to contradictory statements. Obviously, if spin is ignored, one might obtain results in agreement with classical electrodynamics. These are in correspondence with the classical collapse case discussed in [2].

In papers [2], [3] it was proposed the occurrence of a quantum magnetic collapse based in the fact that for a relativistic degenerate fermion gas placed in a very strong external magnetic field, the pressure perpendicular to the field direction vanish for a field strong enough. The degenerate fermion gas is composed either of charged particles as electrons and positrons, or by neutral particles having non-zero magnetic moment, as it is a gas of neutrons in a background of electrons and protons. The collapse may occur since for such a gas: 1) its pressure is exerted anisotropically, having a smaller value perpendicular than along the magnetic field and 2) there are critical values of the magnetic field strength for which the magnetic response of the gas through its magnetization is such that it produces the vanishing of the equatorial pressure of the system, and the outcome would be a transverse collapse of the star. Several authors (i.e. [6], [7]) have referred to our proposal.

We must emphasize that such effect means that for any body, the dimensions transverse to B decrease with increasing B . It is in correspondence with the fact that in the quantum regime a magnetic field leads to a characteristic length $\sqrt{\hbar c/eB}$ which gives a measure of the wave function spread perpendicular to B , and it decreases with increasing B . The *quantum area* $\hbar c/eB$ has the property that the magnetic flux through it leads to a quantum flux $B(\frac{\hbar c}{eB}) = \frac{\hbar c}{e}$ and for increasing B such area decreases. There is a shrinking orthogonal to B which has several consequences in microscopic quantum systems. For instance, in quantum vacuum, for any body placed on a strong magnetic field it is produced a shrinking perpendicular to the field and a stretching along it [8], and recently it has been shown [9] that the Coulomb potential of a charge placed on a strong magnetic field follows an anisotropic law, decreasing away from the charge slower along B and faster across it. For very large magnetic fields the potential is confined to a thin string passing through the charge parallel to B .

Also, a model based on the dynamic description of a local volume of a magnetized self-gravitating Fermi gas starting from a very dense Fermi gas, it has been also shown recently the occurrence of a quantum magnetic collapse along a strong external magnetic field [10].

We would like in the present paper to discuss some points about the role of the energy-momentum tensor on quantum relativistic matter in a magnetic field. We will refer mainly to the electron-positron gas in a magnetic field, as we did in [3], but will refer also to the neutron gas, as a model for neutron stars and we also will comment the magnetized quark matter in the framework of the phenomenological MIT Bag Model to describe Strange Stars [11]-[12].

In the electron-positron case, we show explicitly the vanishing of the squared transverse momentum average, for the particle in the Landau ground state, leading to a vanishing pressure. We are thinking of a model of a white dwarf (or neutron star), in which it is assumed that the dominant pressure is due to the electron (neutron) gas, which can be described as a degenerate quantum gas. For the white dwarf we assume, thus, that there is a nuclei background which compensates the electric charge, but whose thermodynamics is described classically and leads to quantities negligibly small as compared with those of the electron gas. This means that we use the results of relativistic quantum statistics in equilibrium, applied to the study of the properties of an magnetized fermion gas in an external field. We start, thus, from a quantized theory in presence of an external field, as is the problem of electrons in a magnetic field (or either in an atom), where the quanta of the fields involved are the electron-positron and photon fields.

The external fields are, obviously, not considered of quantum nature, and its physical consequences are manifested through their interactions with the quanta, expressed by appropriate terms in the Lagrangian.

We discuss also the thermodynamics for the axially symmetric systems in a magnetic field, having anisotropic properties, and to the fact that material bodies deform to adopt the appropriate shapes to maintain equilibrium, although in some cases the system may become unstable [10] and collapse. Finally we also discuss briefly the form of the electromagnetic stress tensor in relativistic matter, in particular for a self-magnetized system.

II. QUANTUM MECHANICS AND QUANTUM STATISTICS IN EXTERNAL FIELDS

External fields and boundaries have a close analogy. It is usual in the most elementary problems of statistical mechanics, as that of an ideal gas, to assume that there is momentum conservation in molecular collisions. However, the gas is confined to a vessel with which the molecules exchange momentum (and energy), and total conservation of momentum and energy cannot hold unless the vessel is included, as a thermostat. The conservation of energy and momentum is, nevertheless, assumed to take place in the gas *on the average*. Thus, one speak about internal energy and pressures inside the gas, under the previous assumptions. Usually, the energy and momentum of the walls of the vessel do not enter into play in the thermodynamic description of the gas. However, the role of the vessel appears in fixing the external parameter V .

Another point is the inclusion of the classical energy of the electromagnetic field in quantum mechanical calculations. If we calculate the electron energy in the atom, we should *not* add to the usual Dirac electron energy, the term T_{44} coming from the stress tensor produced by the external field of the nucleus. This field is $E = Ze^2/r^2$, and if its square divided by 8π is integrated from some small ϵ to infinity, we would get an expression of form $U = Z^2 e^2/2\epsilon$, which diverges as $\epsilon \rightarrow 0$. This would lead to the expression of *classical* electromagnetic self-energy of the nucleus. There is no reason for adding this energy to the Dirac electron energy eigenvalue. This is nonsense according to standard quantum theory. In a similar way, there is no reason for adding such energy to the quantum statistical average in a problem of many electrons. However, in dealing with the total system (electrons + nuclei), one must consider the problem of self-energy both of electrons and nuclei from the *quantum* point of view. We must consider also conservation of energy and momentum of the whole system.

For the case of electrons moving under the action of an external magnetic field, the problem is similar. The dynamics of the electron is described by its energy eigenvalues and the classical energy of the external magnetic field is a quantity that ultimately corresponds to the energy of the source of the external field (for instance, the energy of the current creating it). As we will see below, all the dynamics of the electrons come from the interaction term of the electron-positron field with the total electromagnetic field (the external field plus the radiation field $A_\mu^{ext} + a_\mu$) in the Lagrangian, and later in the equations of motion, and there is no any reason for adding to the electron energy eigenvalues (and/or to the statistical energy or momentum average) the classical energy and/or momentum of the external field.

Obviously, in considering the conservation laws applied to the total system, it is necessary to take into account the energy and momentum of the sources of the field. These are usually confined to a small region of the total system, and frequently it is possible to describe them classically. In most of the volume of the system, however, the electrons are to be considered as being under the action of an external field, and for them, for instance, the momentum transverse to the field is not conserved.

The thermodynamics of an electron gas in an external field is, in consequence, described also by its interaction term with the electron-positron field in the Lagrangian. For a charged particle in presence of an external magnetic field B (parallel to x_3) the momentum component along the field p_3 is a quantum number, which together with the Landau quantum number $n(= 0, 1, 2, \dots)$ (which do not correspond to momentum eigenvalues), and the spin component along B (the eigenvalues ± 1 of σ_3 , characterizes the set of quantum states. The latter are degenerate with regard to a third quantum number, the orbit's center coordinate $x_0 = p_i/eB$, where $i = 1$ or 2 depending on the gauge choice for the four-potential from which the external field B is derived. For a neutral particle with nonzero magnetic moment q/m , the three components of momentum are observable, but they do not enter symmetrically in the energy spectrum. The component p_3 behaves as in the zero field case, but the transverse momentum $p_\perp = \sqrt{p_1^2 + p_2^2}$ enters in a different way (see below), and for a magnetic dipole moment parallel (antiparallel) to B the contribution is decreased (increased) as B grows. From all this, we conclude that an anisotropy in the quantum dynamics must be reflected in an anisotropy in the thermodynamical properties of the quantum gas in a magnetic field. This leads to the necessary formulation of the statistical physics of the problem taking into account the arising of anisotropic quantities.

The quantum statistics of the problem is made by taking the average of the field operators by using the density matrix operator, which is equivalent to a quantization at finite temperature and nonzero density. The extremely degenerate case corresponds to the zero temperature case. We start from the Lagrangian describing the interaction of the electron-positron and photon fields in presence of an external magnetic field. A background of positive charge, due to the nuclei, is assumed to exist to compensate the electron charge. Its Lagrangian (which usually gives a negligible contribution), as well as the Lagrangian of the subsystem generating the external magnetic field, are not included in the present approximation, which considers only the interaction of the electron gas with a given external magnetic field (assumed as constant and uniform). However, in some cases (for instance, if we consider vector bosons with non-zero magnetic moment [13]), [14], the system may generate self-consistently its magnetic field, and it is not necessary to consider external sources for it. In the present electron case, we have,

$$\mathcal{L} = \bar{\psi}[\gamma_\mu(\partial_\mu - ieA_\mu) - m]\psi + \frac{1}{16\pi}F_{\mu\nu}^2 \quad (2)$$

Here $A_\mu = A_\mu^{ext} + a_\mu$, where $A_\mu^{ext} = Bx_1\delta_{2\mu}$ is the external vector potential in some fixed gauge, and $a_\mu = a_\mu(x)$ is the radiation field (in the case of a relativistic electron in the atom, it is usually taken $A_\mu^{ext} = -\frac{Ze^2}{r}\delta_{0\mu}$, in which it has been fixed also a gauge for the external field). In (2) we exclude terms linear in the fields, whose quantum average is zero. Thus, the pure field term is to be replaced by the sum of the external field tensor squared plus the radiation field tensor squared, i.e. $F_{\mu\nu}^2 \rightarrow F_{\mu\nu}^{ext2} + f_{\mu\nu}^2$. The equations of motion are

$$[\gamma_\mu(\partial_\mu - ieA_\mu) - m]\psi = 0, \quad \frac{\partial f_{\mu\nu}}{\partial x_\nu} = \bar{\psi}\gamma_\mu\psi, \quad (3)$$

where we see that $F_{\mu\nu}^{ext}$ obviously do not enter in the equations of motion: we did not included in our problem the external sources generating physically the external field. We may study, however, the problem of the electron-positron gas in the constant and homogenous external magnetic field consistently, since the interaction of matter with the external field is properly included. Even more, the equations of motion (3) do not contain the contribution from the pure external field term $\frac{1}{16\pi}F_{\mu\nu}^{ext2} = \frac{1}{8\pi}B^2$ (recall that the external field \mathbf{B} remains as only magnetic in all frames of reference moving parallel to it, but the quantity $\frac{1}{8\pi}F_{\mu\nu}^{ext2}$ is a Lorentz invariant), since such term is independent of the fields to be quantized $\bar{\psi}, \psi, a_\mu$ and of the coordinates x_μ . Physically, this means that the particles do not feel the momentum (or pressure) and energy generated by that term, but only the quantities coming from the generalized four-momentum $p_\mu + eA_\mu$, or equivalently, from the interaction with the field $ie\bar{\psi}\gamma_\mu A_\mu\psi$.

III. AVERAGE TRANSVERSE MOMENTUM AND PRESSURE OF THE ELECTRON GAS

The vanishing of the transverse pressure in the Landau ground state can be seen also from quantum mechanical considerations. The electrons (positrons) in an external magnetic field B have energy eigenvalues $E_{n,p_3,\pm 1} = \sqrt{m^2 + p_3^2 + 2eB(n+1/2)} \pm eB$, and its quantum states are described by the spinor wavefunctions $\Phi_n^\pm(x)$ [15]

$$\Phi_{n,p_2,p_3,1}^\pm(x) = \left[\frac{E(p_3, n) \pm m}{2E(p_3, n)} \right]^{1/2} \frac{e^{ip_2x_2 + ip_3x_3}}{2\pi} \begin{bmatrix} \varphi_{n-1}(\xi) \\ 0 \\ \frac{p_3\varphi_{n-1}(\xi)}{m \pm E(p_3, n)} \\ \frac{i(2neB)^{1/2}\varphi_n(\xi)}{m \pm E(p_3, n)} \end{bmatrix} \quad (4)$$

and

$$\Phi_{n,p_2,p_3-1}^\pm(x) = \left[\frac{E(p_3, n) \pm m}{2E(p_3, n)} \right]^{1/2} \frac{e^{ip_2x_2 + ip_3x_3}}{2\pi} \begin{bmatrix} 0 \\ \varphi_n(\xi) \\ -\frac{i(2neB)^{1/2}\varphi_{n-1}(\xi)}{m \pm E(p_3, n)} \\ -\frac{p_3\varphi_n(\xi)}{m \pm E(p_3, n)} \end{bmatrix} \quad (5)$$

where the superscript \pm refers respectively to positive and negative energy solutions, and (4),(5) correspond to the eigenvalues ± 1 of σ_3 , respectively parallel and antiparallel to B . For $n = 0$, only the second (spin down) term contributes. It contains as a factor a term proportional to the Hermite function of zero order $(eB)^{1/4} A_0 e^{-\xi^2/2} e^{ip_2x_2 + ip_3x_3}$ of argument $\xi = (eB)^{1/2}(x + x_0)$, where $x_0 = p_2/eB$ is the coordinate of the orbit's center. The energy eigenvalues are degenerate with regard to x_0 . Expressions (4)-(5) are obviously not eigenfunctions of the transverse momentum operators $i\partial/\partial_{1,2}$. The transverse momentum eigenfunctions would be given by an infinite series in terms of (4),(5) and one can easily check that for the system of non-interacting electrons, if confined to the Landau ground state $n = 0$, the expectation value of its effective momentum operator perpendicular to the field is zero. This is especially interesting because as the magnetic field increases the number of occupied Landau states tends to decrease up to the limit $n = 0$. Since $\partial\varphi_n(\xi)/\partial\xi = \sqrt{n/2}\varphi_{n-1}(\xi) - \sqrt{(n+1)/2}\varphi_{n+1}(\xi)$, and by defining the squared transverse momentum operator as the quantity $[-\partial^2/\partial x_1^2 + \sigma_3 eB/2]$, where we have added the spin contribution $\sigma_3 eB/2$ to the second derivative with regard to x_1 (Obviously, the spin term along the 3-axis we assume as contributing to the momentum perpendicular to B . We did not included the derivative with regard to x_2 since p_2 leads to the coordinate

x_0 of the center of the orbit, and the energy eigenvalue is degenerate with regard to it) one can write the average effective transverse momentum squared p_\perp^2 when the system is in the Landau ground state $n = 0$, $\sigma = -1$, as

$$\int \bar{\Phi}_{0,p_{2,3}}^\pm(x) [-\partial^2/\partial x_1^2 - eB/2] \Phi_{0,p_{2,3}}^\pm(x) dx_1 = 0. \quad (6)$$

The transverse effective momentum in the ground state is thus $p_\perp = \sqrt{E_0^2 - p_3^2 - m^2} = 0$ and the electron gas behaves as a one dimensional (parallel to B) gas and does not exert any pressure perpendicular to B . This effect becomes important for densities of order $\lambda_C^{-3} \sim 10^{30} \text{ cm}^{-3}$ (λ_C is the Compton wavelength) and fields near the critical value $B \sim B_c$, where $B_c \sim m^2 c^3 / e \hbar \sim 10^{13} \text{ Gauss}$. This implies a stretching of the body along the B -field direction and a shrinking perpendicular to it. (The effect of the anomalous magnetic moment of the electron may appear if we consider radiative corrections, i.e., calculations beyond the present tree level).

If one takes into consideration the contribution from quantum vacuum, as it must be, [8], one finds that actually, the pressure resulting in the electron case for Landau quantum number $n = 0$, must be increased in a *negative quantity*, since the transverse pressure of quantum vacuum is negative, which enhances the shrinking effect of the gas.

IV. THE NEUTRON GAS CASE

For the neutron gas, however, the anomalous magnetic moment plays an important role already at the tree level. Due to it, the anisotropy in pressures is similar to the one in the electron-positron gas, leading to a stretching of the body along the magnetic field. As pointed out in [2], the neutron eigenvalues in a magnetic field

$$E_n(p, B, \eta) = \sqrt{p_3^2 + (\sqrt{p_\perp^2 + m_n^2} + \eta q B)^2}, \quad (7)$$

where $\eta = \pm$ and $q = 1.91 M_n$, where M_n is the nuclear magneton. A degenerate neutron gas exhibit the relativistic behavior analog to Pauli paramagnetism. For adequate values of the density and magnetic field, the amount of neutrons with magnetic moment up ($\eta = -1$) is largely greater than those with magnetic moment down ($\eta = +1$). This results from the discussion of the densities of neutrons N_n^\pm with magnetic dipoles up and down ($\eta = \mp 1$), by starting from the Fermi distribution for neutrons in the degenerate case, $n_n(E_n) = \theta(\mu_n - E_n(B, \eta = \mp 1))$. The vanishing of the argument of the $\theta(x)$ functions define the two Fermi surfaces for $\eta = \mp 1$, and the boundaries of the integral for N_n below, which is taken in cylindrical coordinates in momentum space

$$N_n^\pm = \frac{1}{4\pi^2} \sum_{\eta=1,-1} \int_0^{\sqrt{(\mu - \eta q B)^2 - m_n^2}} p_\perp dp_\perp \int_{-p_{3F}}^{p_{3F}} dp_3 \theta(\mu - E_n(p_\perp, dp_3, B, \eta)). \quad (8)$$

Where $p_{3F} = \sqrt{\mu^2 - (\sqrt{p_\perp^2 + m_n^2} + \eta q B)^2}$. We observe that for $\mu - qB \rightarrow m_n$ the upper limit of integration in (8) tends to 0, which means that N^- also tends to zero. We shall consider in what follows B large enough to satisfy approximately these conditions, so that the term N^+ has the main contribution to N . The Fermi surface for $\eta = -1$ is defined by $\mu_n - E_n(B, \eta = -1) = 0$. The effective Fermi transverse momentum squared is

$$p_{F\perp eff}^2 = \mu_n^2 - p_{F3}^2 - m_n^2 = \left(\sqrt{p_{F\perp}^2 + m_n^2} - qB \right)^2 - m_n^2. \quad (9)$$

The second term of (9) decreases with increasing B , and may even vanish for some critical fields. For instance, if $B \ll B_{cn}$, where $B_{cn} = m_n/q \sim 10^{20} \text{ G}$, then $q^2 B^2 \ll 2qBm_n$. In such case the vanishing of $p_{F\perp eff}$ is guaranteed if $p_{F\perp} \sim \sqrt{2qBm_n}$. Thus, the neutron gas statistical behavior under these physical conditions is equivalent to a one-dimensional gas, with effective spectrum $E_n(p, B) = \sqrt{p_3^2 + m_n^2}$. The transverse pressure vanishes. For $p_{F\perp} \sim 10^{-1.5} m_n$, $qB/m_n \sim 10^{-3}$, which means fields of order $B \sim 10^{17} \text{ G}$.

V. MAGNETIZED STRANGE QUARK MATTER

Strange quark matter (SQM), which means to have quarks u d and s and electrons, could be studied using the MIT bag model [11]. In that model, confinement is guaranteed by the bag and quarks are considered as a Fermi gas of

noninteracting particles. Under these assumptions, it is possible to study the thermodynamical properties of a quark gas in a strong magnetic field. In our study the anomalous magnetic moment (AMM) is included, which means that Pauli paramagnetism is taken into account besides Landau diamagnetism, given by the presence of Landau levels. In that case the energy spectrum lose its degeneracy. The thermodynamical quantities depends on two sums: one over Landau levels and other over the two orientations of the spin parallel or antiparallel to the magnetic field. As the particles (quarks) have positive or negative AMM, they have different preferences in the spin orientation with respect to the magnetic field. This has important consequences in the EOS of the system. The most relevant comes from the energy ground state, which depends on the strength of the magnetic field and it could be zero (cf. Eq. (13)).

$$E_{i,n}^\eta = \sqrt{p_3^2 + m_i^2 \left(\sqrt{\frac{B}{B_i^c}} (2n+1-\eta) + 1 - \eta Q_i B \right)^2}, \quad (10)$$

with $B_i^c = \frac{m_i^2}{|e_i|}$ $i = (e, u, d, s)$, e_i and m_i denote the charges and the masses of the particles, respectively. The quantities Q_i are the corresponding AMM of the particles,

$$\begin{aligned} Q_e &= 0.00116\mu_B, & Q_u &= 1.85\mu_N, \\ Q_d &= -0.97\mu_N, & Q_s &= -0.58\mu_N, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \mu_B &= \frac{e}{2m_e} \simeq 5.79 \times 10^{-15} \text{ MeV/G}, \\ \mu_N &= \frac{e}{2m_p} \simeq 3.15 \times 10^{-18} \text{ MeV/G}. \end{aligned} \quad (12)$$

$m_u = m_d = 5 \text{ MeV}$ and $m_s = 150 \text{ MeV}$ for the light quark masses. The magnitudes of the so-called critical fields B_i^c (when particle's cyclotron energy is comparable to its rest mass) are $B_e^c = 4.4 \times 10^{13} \text{ G}$, $B_u^c = 6.3 \times 10^{16} \text{ G}$, $B_d^c = 1.3 \times 10^{16} \text{ G}$ and $B_s^c = 1.1 \times 10^{19} \text{ G}$.

It can be seen from the spectra (10) that, besides of the quantization of their orbits in the plane perpendicular to the magnetic field, charged particles with AMM undergo the splitting of the energy levels with the corresponding disappearance of the spectrum degeneracy. For the non-anomalous case, $Q_i = 0$, the ground state energy is independent of the magnetic field strength and the magnetic field only quantizes the kinetic energy perpendicular to the field similar to that of the electron case. In this situation, the energy is degenerate for Landau levels higher than zero. States with spin parallel or antiparallel to the magnetic orientations ($\eta = \pm 1$) have the same energy. However, the anomalous case, $Q_i \neq 0$, removes this degeneracy. In the latter case, the rest energy of the particles depends on the magnetic field strength. The ground state energy is

$$E_{i,0} = m_i (1 - y_i B). \quad (13)$$

The above equation leads to the appearance of a threshold value for the magnetic field at which the effective mass vanishes, $m_i \sim |Q_i|B$. The thresholds of the field, $B_i^s = m_i/Q_i$, for all the constituents of the SQM are given by

$$\begin{aligned} B_e^s &= 7.6 \times 10^{16} \text{ G}, & B_u^s &= 8.6 \times 10^{17} \text{ G}, \\ B_d^s &= 1.6 \times 10^{18} \text{ G}, & B_s^s &= 8.2 \times 10^{19} \text{ G}, \end{aligned} \quad (14)$$

that are smaller than the ones obtained when the classical AMM contribution is considered [16].

The expression (13) suggests that the energy of the particles becomes smaller than that of the antiparticles, which might be manifested in the creation of pairs. As spontaneous pair creation in a magnetic field seems to be forbidden, for individual particles, the correct meaning of this "critical" field is that it corresponds to an upper bound.

In the SQM scenario all the constituents interact with the magnetic field and are obliged to satisfy the equilibrium conditions. Under such constraints, it turns out that the dominant threshold field comes from u quarks, thus leading to the upper bound $B \lesssim 8.6 \times 10^{17} \text{ G}$. This result has an important astrophysical consequence, since the bound for SQM can be also extrapolated to the SQS scenario. If SQS exist, the maximum magnetic field strength that they could support would be around the above bound, i.e. 10^{18} G .

The density of particles, defined as $N = \sum_i N_i$ with $N_i = \frac{\partial \Omega_i}{\partial \mu_i}$ gives

$$N_i^\pm = \frac{1}{4\pi^2} \sum_n^{nmax} \sum_{\eta=1,-1} \int_0^{\sqrt{(\mu - \eta Q_i B)^2 - eB(2n+1-\eta) - m_i^2}} \int_{-p_{3F}}^{p_{3F}} dp_3 \theta(\mu_i - E_i(p_\perp, dp_3, B, \eta)). \quad (15)$$

with

$$p_{F\perp,i}^2 = \mu_i^2 - p_{F3}^2 - m_i^2 = \left(\sqrt{e_i B(2n+1-\eta) + m_i^2} - \eta Q_i B \right)^2 - m_i^2, \quad (16)$$

The presence of AMM produces that $p_{F\perp eff i}$ decreases with the increasing of magnetic field. For $n = 0$ we have that

$$p_{\perp} = BQ_i |BQ_i - 2m_i|$$

which would be zero for $B = 2m_i/Q_i$.

Nevertheless, for the SQM in β -equilibrium never is reached the condition $p_{\perp} = 0$. It is due to the restriction imposing by β -equilibrium that makes that appear an upper bound for the magnetic field that is one half of this one that does $p_{\perp} = 0$. This condition is reached because the chemical potential of electrons involved in SQM becomes negative which is lacking of physical sense. If AMM is not take into account we obtain the condition $p_{\perp} = 0$ when $n = 0$ for fields around 10^{19} G [11].

From the quantum statistical point of view the lowest energy states with AMM contain important physical consequences: for particles with mass m_i and anomalous magnetic moment Q_i , the magnetic field has a critical value given by the expression $B_i^s \sim m_i/|Q_i|$. For SQM in β -equilibrium it turns out that the dominant threshold field comes from u quarks, thus leading to the upper bound $B \lesssim 8.6 \times 10^{17}$ G [12].

VI. THERMODYNAMICS FOR ANISOTROPIC SYSTEMS

In this section we shall assume that we are in a reference frame in which the center of mass of the system we are considering is at rest. Quantum statistics is currently formulated for isotropic systems. In presence of an external electromagnetic field, the entropy S and the energy U , which are basic scalars, are dependent on the tensor field $\mathcal{F}_{\mu\nu}$. The external field breaks the spatial isotropy and this is reflected in the quantum mechanical observables. In standard statistical mechanics, in which the system is assumed as isotropic, the usual interpretation of thermodynamic quantities is through the relation $\Omega = -PV$, where $-P = \Omega_V = -\beta^{-1} \ln \mathcal{Z}$ (we are naming $\Omega_V = \Omega/V$). Here we assume that \mathcal{Z} is built from a density matrix $\rho = e^{-\beta \int d^3x (\mathcal{H} - \mu N)}$ where the integral is taken over all space. There is one external parameter V , and one conjugate variable P . Hidden in this isotropic formulation is the idea of the elementary work $\delta W = f_i dx_i$ done by a generalized force f_i in the direction of the elementary displacement dx_i . This generalized force comes from the product of the isotropic pressure tensor $P_{ij} = p\delta_{ij}$ by the surface element dS_j , e.g. $f_i = P_{ij} d\sigma_j = p dS_i dx_i = p dV$.

In the anisotropic case, P_{ij} even if having a diagonal form, have in general different eigenvalues. The elementary work comes from the product of the force $f_i = P_{ij} d\sigma_j$ by the elementary displacement dx_i . One gets $\delta W = f_i dx_i = P_{ij} d\sigma_j dx_i$. Here the product of the differentials behaves formally as a "volume pseudo-tensor" $dV_{ji} = d\sigma_j dx_i$ (This is restricted to three dimensional space. Our considerations strictly apply in the frame where the body is at rest). Actually, the entropy density and the particle density may be also anisotropic and given by pseudo-tensors s_{ij} , n_{ij} , so that $dS = s_{ij} dV_{ji}$, $dN = n_{ij} dV_{ji}$. The differential of the energy can be written as always, by the sum of the elementary heat $\delta Q = T dS$ plus the elementary work done by the system $\delta W = -P_{ij} dS_j dx_i$.

$$dU = T dS + \mu dN - P_{ij} dV_{ji}. \quad (17)$$

From (17), one can see that dU may have a different expression for elementary changes in the volume of the system taken along different directions in space.

Here arises a very important point of procedure. For isotropic systems of volume V , the thermodynamic potential $\Omega = -PV$ is the equation of state arising from the evaluation of Ω in terms of the Hamiltonian operator, temperature and chemical potential. There is only one pressure, since the energy-momentum tensor is spatially isotropic $\mathcal{T}_{\mu\nu}^i = P\delta_{\mu\nu} - (P + u)\delta_{4\mu}\delta_{\nu 4}$. However, once we have anisotropic pressures, such relation is not enough.

When an external magnetic field is present, one can write in terms of the thermodynamic potential Ω_V [17],

$$\mathcal{T}_{\mu\nu} = (T\partial\Omega_V/\partial T + \mu\partial\Omega_V/\partial\mu)\delta_{4\mu}\delta_{\nu 4} + 4F_{\mu\rho}F_{\nu\rho}\partial\Omega_V/\partial F^2 - \delta_{\mu\nu}\Omega_V, \quad (18)$$

This tensor reduces to the isotropic expression $\mathcal{T}_{\mu\nu}^i$ for $F_{\mu\rho} = 0$.

VII. SOME THERMODYNAMICAL RELATIONS

For simplicity, we shall assume that shearing stresses are absent. From the thermodynamical point of view, the new element is that there are two kind of external parameters, the volume and the external field, which can be defined either by the electromagnetic field tensor $\mathcal{F}_{\mu\nu}$ or by the three dimensional vectors \mathbf{E} , \mathbf{B} , and as conjugate parameters, the pressures and the electric and magnetic moments P , \mathcal{M} . For instance, in the case of a constant homogeneous magnetic field one has $\Omega_V = f(\mu, \beta, B)$, and from the relation $U = TS + \mu N + \Omega$ [18] one can write in terms of the extensive variables S, N, V_{ij} , and the intensive variable B ,

$$dU = TdS + \mu dN + (\partial\Omega/\partial V_{ij})dV_{ij} + (\partial\Omega/\partial B)dB. \quad (19)$$

where $\partial\Omega/\partial B = V\partial\Omega_V/\partial B = -V\mathcal{M}$, \mathcal{M} is the magnetization and $V\mathcal{M}$ the magnetic moment. We name again $\partial\Omega/\partial V = -P$ "the pressure", although care must be exercised here. The interpretation of this term as the pressure is free from ambiguities in the isotropic case, where the spatial diagonal components of the energy-momentum tensor are equal. In presence of an external field, this is not the situation. Let us consider the elementary work done for an increase in the volume dV_\perp in the direction orthogonal to B . We have

$$\delta W_\perp = -(p - \mathcal{M}B)dV_\perp. \quad (20)$$

However, along the field, it is $\delta W_\parallel = -pdV_\parallel$. We have different results depending on the direction in space. We will concentrate in the pressures exerted by the electron gas. We obtain from (17) or (18) different expressions for the pressure for directions parallel and perpendicular to the magnetic field,

$$P_3 = -\Omega_V, \quad P_\perp = -\Omega_V - B\mathcal{M}. \quad (21)$$

The fourth component of the tensor is the expression for the energy density

$$u = Ts + \mu\rho - \Omega_V, \quad (22)$$

where $s = -\partial\Omega_V/\partial T$, $\rho = -\partial\Omega_V/\partial\mu$ are respectively the entropy and particle number densities.

Concerning the thermodynamical properties of the magnetized gases, from the definition of $\mathcal{M}(B)$ we get $\Omega = -\int \mathcal{M}dB - P_0$, P_0 being the pure mechanical pressure, i.e. the part of the pressure term independent of the field. One can write

$$dP_0 = dP_3 - \mathcal{M}dB. \quad (23)$$

From (22) and (19) we can write for the differential of internal energy U when the volume V changes along B , i.e., along the 3-rd axis as $dV = S_\perp dx_3$, where (S_\perp is the basis of the volume V),

$$dU = TdS + \mu dN - P_3 dV - \mathbf{M} \cdot d\mathbf{B}. \quad (24)$$

Thus, if $dS = dN = 0$ and B is kept constant, the change in internal energy is the work $\delta W_3 = -P_3 dV = -F_3 dx_3$, where $F_3 = P_3 dS_\perp$ is the generalized force along x_3 . For a displacement perpendicular to the field one expects that although B and \mathcal{M} be kept constant, \mathbf{M} must change. To account for it, we may define another thermodynamic function by means of a Legendre transformation. By adding $d(\mathbf{M} \cdot \mathbf{B})$ to (24) one gets the differential of the "extended energy" U_0 as

$$\begin{aligned} dU_0 &= dU + d(\mathbf{M} \cdot \mathbf{B}) = TdS + \mu dN - P_3 dV + \mathbf{B} \cdot d\mathbf{M} \\ &= TdS + \mu dN - P_\perp dV. \end{aligned} \quad (25)$$

Also if $dS = dN = dB = 0$, the last term in (26) is the work done when $dV = S_3 dx_\perp$, where S_3 is one section of V containing the 3-rd axis. In other words $dU_0 = \delta W_\perp = -P_\perp dV = -F_\perp dx_\perp$, where $F_\perp = P_\perp S_3$ is the generalized force perpendicular to B . We observe that (26) has the usual form of the differential of the internal energy in the

zero field case written in terms of the P_\perp pressure. Note that for $P_3 \gg \mathcal{M}B$, the $P_\perp \sim P_3$ and U_0 corresponds to the usual expression for the internal energy in absence of the field, whereas $U = U_0 - \mathbf{M} \cdot \mathbf{B}$ expresses the internal energy after including the interaction of the system, as a dipole \mathbf{M} , with the external field \mathbf{B} .

Finally, by adding $d(P_3 V)$ to dU one has for the differential of the enthalpy $H = U + P_3 V$

$$dH = TdS + \mu dN - V(dP_3 - \mathcal{M}dB). \quad (26)$$

Thus, if N is constant and the pure mechanical pressure is constant, $dP_0 = 0$, dH is equal to the heat absorbed by the system.

VIII. DEFORMATIONS OF BODIES DUE TO ANISOTROPY

We must discuss the physical consequences of the anisotropy in pressures. Thinking about a star as a model, at any point the gas pressure is counterbalanced by the pressure due to the gravitational attraction of the star mass. Let us name this pressure P_g . The equilibrium of the pressures (21) with P_g leads to

$$P_3 = P_g, \quad P_\perp = P'_g. \quad (27)$$

The second equation is to be understood as meaning that $P_3 = P'_g + B\mathcal{M}$. This problem bears some parallelism with the rotating body in a gravitational field: in that case the effective gravitational field perpendicular to the axis of rotation is decreased by the centrifugal force. Here the situation is the opposite, as the gravitational pressure perpendicular to B is increased in the amount $B\mathcal{M}$. The outcome is a prolate isobar surface, or in other words, the body stretches along B and shrinks perpendicular to it, like a cigar. Also, from (21) and according to [3], for all the electrons in the Landau ground state $n = 0$ (which does not contradict Pauli's Principle, since there is a degeneracy with regard to an extra quantum number, the orbit's center), $-\Omega_V = B\mathcal{M}$ and $p_\perp = 0$ which suggests the occurrence of a collapse of the system perpendicular to \mathbf{B} .

IX. THE ELECTROMAGNETIC STRESS TENSOR IN RELATIVISTIC MATTER AND SELF-MAGNETIZATION

By writing $\mathcal{M} = (H - B)/4\pi$, one may get formally $\Omega = -\frac{1}{8\pi}B^2 + \frac{1}{4\pi} \int H dB - p_0$. We recall that as $\Omega \equiv F - G$, the last expression is consistent with what would be obtained in the classical non-relativistic case [18], where $F = F_0 + \int H dB/4\pi$ is the Helmholtz free energy and $G = F + \int \mathcal{M} dB + p_0 = G_0 + B^2/8\pi$ as the Gibbs free energy. Due to our definition of \mathcal{M} , our last term is given in terms of B and not in terms of H . These expressions are useful for obtaining an expression for the stress tensor in a medium. We would like to note that there is no a unique expression accepted by all authors [19] for it in terms of the fields B and H . We want to state here the relation we have obtained among the energy-momentum tensors involved in our previous formulae, which is the sum of the Maxwell tensor for the field B plus the Minkowski tensor for a nonlinear media, involving both B and H . As we have also $-\Omega - B\mathcal{M} = -\frac{1}{8\pi}B^2 + \frac{1}{4\pi} \int B dH + p_0$, we observe then that one can write \mathcal{T}_{ij} as

$$\mathcal{T}_{ij} = p_0 \delta_{ij} + \mathcal{S}(B)_{ij} - \mathcal{T}_{ij}^M(B, H), \quad (28)$$

where $\mathcal{S}(B)_{ij} = \frac{1}{4\pi}[B_i B_j - \frac{1}{2}(B^2)\delta_{ij}]$ is the Maxwell stress tensor for the microscopic field B , and $\mathcal{T}_{ij}^M(B) = \frac{1}{4\pi}[H_i B_j - (\int B dH)\delta_{ij}]$ is the Minkowski tensor for nonlinear media, which reduces to the usual expression [19] in the linear case. The equation (28) expresses the way the stress tensors appear in our problem.

If $H = 0$, then $B = 4\pi\mathcal{M}$ and $\mathcal{T}_{ij} = p_0 \delta_{ij} + \mathcal{S}(B)_{ij}$. This gives the total pressure when the magnetic field is maintained self-consistently, for instance, in a ferromagnetic body.

This case deserves some comments. For the system below the ferromagnetic phase transition (where it is yet paramagnetic), we start from the thermodynamic potential density, $\Omega_V = -P_0 - \int \mathcal{M} dB$. We assume that it must be an even function of B , and conclude that \mathcal{M} must be an odd function of B , which we write as $\mathcal{M} = B/4\pi - aB^3 + \dots$. We wrote the second term as negative, since we consider the system below the condition of self-magnetization. The next powers of B are neglected since they are expected to be small.

The extremum of Ω_V with regard to B is determined by the term $\int \mathcal{M} dB$. We have $d\Omega/dB = -B/4\pi + 4aB^3 = 0$ which for $B > 0$ leads to $B^2 = 1/(4a\pi)$. This solution leads to a minimum for $\Omega_V = \Omega_V(B)$, and for it $\mathcal{M} = B/4\pi$:

the minimum of Ω_V corresponds to a self-magnetization for which $\Omega_V = -P_0 - B^2/8\pi$. Thus the ferromagnetic phase transition leads to a spontaneous symmetry breaking, leading Ω_V to reach a minimum value for nonzero B . (for classical and diamagnetic systems, as $\mathcal{M} < 0$, the minimum is achieved for $B = 0$). Obviously, a is a function of temperature and the condition of self-magnetization defines the Curie temperature T_c . The present discussion is especially relevant to vector Boson systems having a magnetic moment, which is the case discussed in refs. [13] and [14], in which self-magnetization occurs.

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